

BÀI 9 : Tính :

$$1) \lim \frac{1+3+3^2+\dots+3^n}{1+4+4^2+\dots+4^n}$$

$$2) \lim \frac{2+2^2+2^3+\dots+2^n}{3+3^2+3^3+\dots+3^n}$$

$$3) \lim \left(\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)} \right)$$

$$4) \lim \frac{1+2+2^2+\dots+2^n}{1+5+5^2+\dots+5^n}$$

$$5) \lim \frac{1+4+7+\dots+(3n+1)}{n^2+2n+1}$$

$$6) \lim \left[\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{(n-1)n} \right]$$

$$7) \lim \frac{1+\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^n}}{1+\frac{1}{5}+\frac{1}{5^2}+\dots+\frac{1}{5^n}}$$

$$8) \lim \left(1 - \frac{1}{2^2} \right) \left(1 - \frac{1}{3^2} \right) \dots \left(1 - \frac{1}{n^2} \right)$$

$$9) \lim \left(\frac{2^3-1}{2^3+1} \cdot \frac{3^3-1}{3^3+1} \cdots \frac{n^3-1}{n^3+1} \right)$$

ĐS : 1) 0 ; 2) 0 ; 3) 1 ; 4) 0 ; 5) 3/2 ; 6) 1 ; 7) 8/5 ; 8) 1/2 ; 9) 2/3.

▪ Hướng dẫn :

1) $\lim \frac{1+3+3^2+\dots+3^n}{1+4+4^2+\dots+4^n}$

Áp dụng công thức tính tổng của cấp số nhân : $S_n = u_1 \frac{1-q^n}{1-q}$ hay $S_n = u_1 \frac{q^n-1}{q-1}$ ($q \neq 1$)

$$1+3+3^2+\dots+3^n = 1 \cdot \frac{1-3^{n+1}}{1-3} = \frac{1}{2} (3^{n+1}-1) = \frac{1}{2} (3 \cdot 3^n - 1)$$

$$\text{và } 1+4+4^2+\dots+4^n = 1 \cdot \frac{1-4^{n+1}}{1-4} = \frac{1}{3} (4^{n+1}-1) = \frac{1}{3} (4 \cdot 4^n - 1)$$

$$\text{Do đó } \lim \frac{1+3+3^2+\dots+3^n}{1+4+4^2+\dots+4^n} = \lim \frac{3(3 \cdot 3^n - 1)}{2(4 \cdot 4^n - 1)} = \lim \frac{3 \left[3 \left(\frac{3}{4} \right)^n - \left(\frac{1}{4} \right)^n \right]}{4 \left[4 - \left(\frac{1}{4} \right)^n \right]} = 0$$

2) $\lim \frac{2+2^2+2^3+\dots+2^n}{3+3^2+3^3+\dots+3^n}$

Áp dụng công thức tính tổng của cấp số nhân : $S_n = u_1 \frac{1-q^n}{1-q}$ hay $S_n = u_1 \frac{q^n-1}{q-1}$ ($q \neq 1$)

$$2+2^2+2^3+\dots+2^n = 2 \cdot \frac{1-2^n}{1-2} = -2(1-2^n) = 2(2^n - 1)$$

$$\text{và } 3+3^2+3^3+\dots+3^n = 3 \cdot \frac{1-3^n}{1-3} = -6(1-3^n) = 6(3^n - 1)$$

$$\lim \frac{2+2^2+2^3+\dots+2^n}{3+3^2+3^3+\dots+3^n} = \lim \frac{2(2^n - 1)}{6(3^n - 1)} = \lim \frac{2 \left[\left(\frac{2}{3} \right)^n - \left(\frac{1}{3} \right)^n \right]}{6 \left[1 - \left(\frac{1}{3} \right)^n \right]} = 0$$

3) $\lim \left(\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)} \right)$

Ta có: $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

$$\lim \left(\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)} \right) = \lim \left(\frac{n}{n+1} \right) = 1$$

4) $\lim \frac{1+2+2^2+\dots+2^n}{1+5+5^2+\dots+5^n}$ ĐS: 0

Áp dụng công thức tính tổng của cấp số nhân: $S_n = u_1 \frac{1-q^n}{1-q}$ hay $S_n = u_1 \frac{q^n-1}{q-1}$ ($q \neq 1$)

$$1+2+2^2+\dots+2^n = 1 \cdot \frac{1-2^{n+1}}{1-2} = 2^{n+1}-1 = 2 \cdot 2^n - 1$$

$$\text{và } 1+5+5^2+\dots+5^n = 1 \cdot \frac{1-5^{n+1}}{1-5} = \frac{1}{4}(5^{n+1}-1) = \frac{1}{4}(5 \cdot 5^n - 1)$$

$$\text{Do đó } \lim \frac{1+2+2^2+\dots+2^n}{1+5+5^2+\dots+5^n} = \lim \frac{4(2 \cdot 2^n - 1)}{5 \cdot 5^n - 1} = \lim \frac{4 \left[2 \left(\frac{2}{5} \right)^n - \left(\frac{1}{5} \right)^n \right]}{5 - \left(\frac{1}{5} \right)^n} = 0$$

5) $\lim \frac{1+4+7+\dots+(3n+1)}{n^2+2n+1}$ ĐS: $\frac{3}{2}$

Xét dãy 1, 4, 7, ..., $(3n+1)$ là một dãy cấp số cộng có $u_1 = 1$; $d = 3$.

Số hạng $u_x = 3n+1 = u_1 + (x-1)d \Leftrightarrow 3n+1 = 1 + (x-1)3 \Leftrightarrow 3n+1 = 1 + 3x - 3 \Leftrightarrow x = n+1$.

Áp dụng công thức tính tổng của cấp số cộng: $S_n = \frac{n}{2}(u_1 + u_n)$ Hoặc: $S_n = \frac{n}{2}[2u_1 + (n-1)d]$

$$\Rightarrow 1+4+7+\dots+(3n+1) = S_{n+1} = \frac{n+1}{2}(1+3n+1) = \frac{(n+1)(3n+2)}{2}$$

$$\text{Vậy } \lim \frac{1+4+7+\dots+(3n+1)}{n^2+2n+1} = \lim \frac{(n+1)(3n+2)}{2(n^2+2n+1)} = \frac{3}{2}$$

6) $\lim \left[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{(n-1)n} \right]$ ĐS: 1

$$\text{Ta có: } \frac{1}{1 \cdot 2} = \frac{1}{1} - \frac{1}{2}; \frac{1}{2 \cdot 3} = \frac{1}{2} - \frac{1}{3}; \frac{1}{3 \cdot 4} = \frac{1}{3} - \frac{1}{4}; \dots; \frac{1}{(n-1)n} = \frac{1}{n-1} - \frac{1}{n} \Rightarrow \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{(n-1)n} = 1 - \frac{1}{n}$$

$$\text{Do đó: } \lim u_n = \lim \left(1 - \frac{1}{n} \right) = 1 - 0 = 1$$

7) $\lim \frac{1+\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^n}}{1+\frac{1}{5}+\frac{1}{5^2}+\dots+\frac{1}{5^n}}$ ĐS: $\frac{8}{5}$

Áp dụng công thức tính tổng của cấp số nhân: $S_n = u_1 \frac{1-q^n}{1-q}$ hay $S_n = u_1 \frac{q^n-1}{q-1}$ ($q \neq 1$)

$$\text{Ta có: } 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} = 1 \cdot \frac{1 - \left(\frac{1}{2} \right)^{n+1}}{1 - \frac{1}{2}} = 2 \left(1 - \left(\frac{1}{2} \right)^{n+1} \right)$$

$$1 + \frac{1}{5} + \frac{1}{5^2} + \dots + \frac{1}{5^n} = 1 \cdot \frac{1 - \left(\frac{1}{5} \right)^{n+1}}{1 - \frac{1}{5}} = \frac{5}{4} \left(1 - \left(\frac{1}{5} \right)^{n+1} \right) \text{ nên } \lim u_n = \frac{8}{5}$$

8) $\lim \left(1 - \frac{1}{2^2} \right) \left(1 - \frac{1}{3^2} \right) \dots \left(1 - \frac{1}{n^2} \right)$ ĐS: $\frac{1}{2}$

$$\text{Với } k \geq 2 \text{ ta có: } 1 - \frac{1}{k^2} = \frac{k^2 - 1}{k^2} = \frac{(k-1)(k+1)}{k^2}$$

$$\text{Do đó } u_n = \frac{1}{2^2} \cdot \frac{2}{3^2} \cdot \frac{3}{4^2} \cdots \frac{(n-3)(n-1)}{(n-2)^2} \cdot \frac{(n-2)n}{(n-1)^2} \cdot \frac{(n-1)(n+1)}{n^2} = \frac{1}{2} \cdot \frac{n+1}{n} = \frac{1}{2} \left(1 + \frac{1}{n}\right).$$

Vậy $\lim u_n = \frac{1}{2}$.

9) $\lim \left(\frac{2^3 - 1}{2^3 + 1} \cdot \frac{3^3 - 1}{3^3 + 1} \cdots \frac{n^3 - 1}{n^3 + 1} \right)$ DS: $\frac{2}{3}$

$$\text{Với } k \geq 2 \text{ ta có: } \frac{k^3 - 1}{k^3 + 1} = \frac{(k-1)(k^2 + k + 1)}{(k+1)(k^2 - k + 1)} = \frac{(k-1)(k^2 + k + 1)}{(k+1)[(k-1)^2 + (k-1) + 1]}$$

$$\frac{2^3 - 1}{2^3 + 1} = \frac{(2-1)(2^2 + 2 + 1)}{(2+1)(2^2 - 2 + 1)} = \frac{(2-1)(2^2 + 2 + 1)}{(2+1)[(2-1)^2 + (2-1) + 1]} = \frac{7}{3.3}$$

$$\frac{3^3 - 1}{3^3 + 1} = \frac{(3-1)(3^2 + 3 + 1)}{(3+1)(3^2 - 3 + 1)} = \frac{(3-1)(3^2 + 3 + 1)}{(3+1)[(3-1)^2 + (3-1) + 1]} = \frac{2.13}{4.7}$$

$$\frac{4^3 - 1}{4^3 + 1} = \frac{(4-1)(4^2 + 4 + 1)}{(4+1)(4^2 - 4 + 1)} = \frac{(4-1)(4^2 + 4 + 1)}{(4+1)[(4-1)^2 + (4-1) + 1]} = \frac{3.21}{5.13}$$

$$\frac{5^3 - 1}{5^3 + 1} = \frac{(5-1)(5^2 + 5 + 1)}{(5+1)(5^2 - 5 + 1)} = \frac{(5-1)(5^2 + 5 + 1)}{(5+1)[(5-1)^2 + (5-1) + 1]} = \frac{4.31}{6.21}$$

$$\frac{n^3 - 1}{n^3 + 1} = \frac{(n-1)(n^2 + n + 1)}{(n+1)(n^2 - n + 1)} = \frac{(n-1)(n^2 + n + 1)}{(n+1)[(n-1)^2 + (n-1) + 1]} = \frac{(n-1)(n^2 + n + 1)}{(n+1)[n^2 - n + 1]}$$

$$\frac{(n-1)^3 - 1}{(n-1)^3 + 1} = \frac{(n-1-1)(n-1)^2 + (n-1) + 1}{(n-1+1)(n-1)^2 - (n-1) + 1} = \frac{(n-2)(n^2 - n + 1)}{n(n^2 - 3n + 3)}$$

$$\frac{2^3 - 1}{2^3 + 1} \cdot \frac{3^3 - 1}{3^3 + 1} \cdots \frac{n^3 - 1}{n^3 + 1} = \frac{7}{3.3} \cdot \frac{2.13}{4.7} \cdot \frac{3.21}{5.13} \cdots \frac{4.31}{6.21} \cdots \frac{(n-2)(n^2 - n + 1)}{n(n^2 - 3n + 3)} \cdot \frac{(n-1)(n^2 + n + 1)}{(n+1)[n^2 - n + 1]}$$

Áp dụng, tính gọn ta được $u_n = \frac{2}{3} \cdot \frac{n^2 + n + 1}{n(n+1)} = \frac{2}{3} \cdot \frac{1 + \frac{1}{n} + \frac{1}{n^2}}{\left(1 + \frac{1}{n}\right)}$ nên $\lim u_n = \frac{2}{3}$